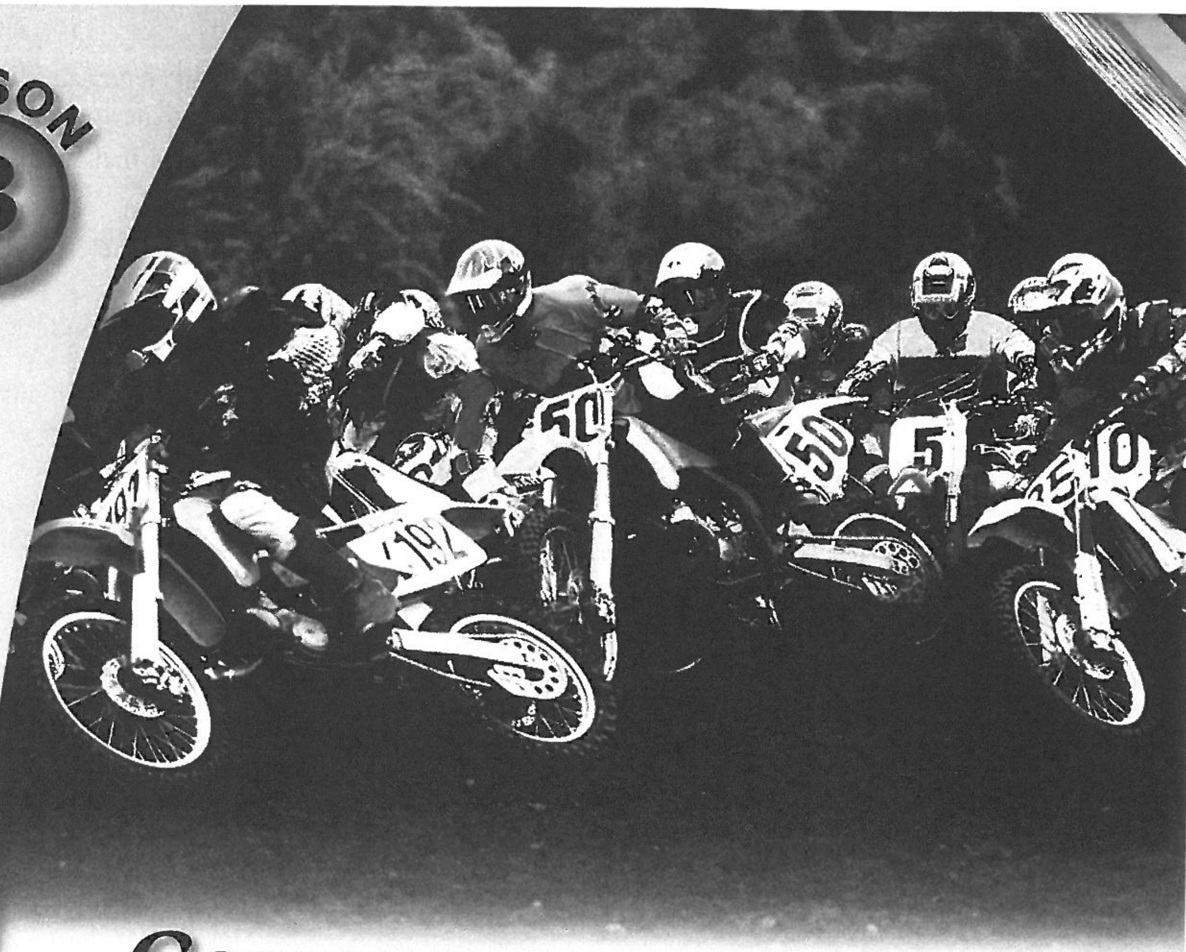
LESSON 3



Common Logarithms and Exponential Equations

Have you ever had someone tell you that you were speaking too loudly (or too softly) or that the volume on a television was turned up too high (or down too low)? While sensitivity to noise can vary from person to person, in general, people can hear sounds over an incredible range of loudness.

Sound intensity is measured in physical units of watts per square centimeter. But the loudness is typically reported in units called decibels. The next table shows intensity values for a variety of familiar sounds and the related number of decibels—as measured at common distances from the sources.

Sound Intensity (in watts/cm^2)	Noise Source	Relative Intensity (in decibels)
10^3	Military rifle	150
10^2	Jet plane (30 meters away)	140
10^1	[Level at which sound is painful]	130
10^0	Amplified rock music	120
10^{-1}	Power tools	110
10^{-2}	Noisy kitchen	100
10^{-3}	Heavy traffic	90
10^{-4}	Traffic noise in a small car	80
10^{-5}	Vacuum cleaner	70
10^{-6}	Normal conversation	60
10^{-7}	Average home	50
10^{-8}	Quiet conversation	40
10^{-9}	Soft whisper	30
10^{-10}	Quiet living room	20
10^{-11}	Quiet recording studio	10
10^{-12}	[Barely audible]	0

Source: *Real-Life Math: Everyday Use of Mathematical Concepts,* Evan Glazer and John McConnell, 2002.

Think About This Situation

Study the sound intensity values, sources, and decibel ratings given in the table.

a Are the intensity and decibel numbers in the order of loudness that you would expect for the different familiar sources?

b What pattern do you see relating the sound intensity values (watts/cm^2) and the decibel numbers?

In this lesson, you will learn about *logarithms*—the mathematical idea us to express sound intensity in decibels and to solve a variety of problems related to exponential functions and equations.

Investigation 1 How Loud is Too Loud?

Your analysis of the sound intensity data might have suggested several different algorithms for converting watts per square centimeter into decibels. For example,

$$\text{if the intensity of a sound is } 10^x \frac{\text{watts}}{\text{cm}^2},$$
$$\text{its loudness in decibels is } 10x + 120.$$

The key to discovery of this conversion rule is the fact that all sound intensities were written as powers of 10. What would you have done if the sound intensity readings had been written as numbers like 3.45 $\frac{\text{watts}}{\text{cm}^2}$ or 0.0023 $\frac{\text{watts}}{\text{cm}^2}$?

As you work on problems of this investigation, look for an answer to this question:

How can any positive number be expressed as a power of 10?

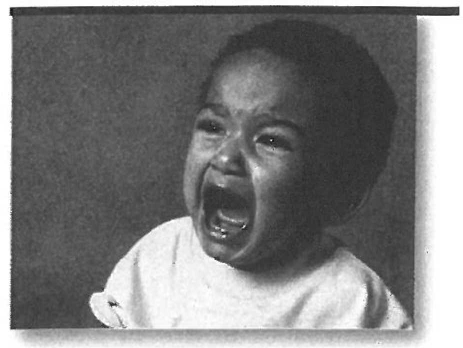
1 Express each of the numbers in Parts a–i as accurately as possible as a power of 10. You can find exact values for some of the required exponents by thinking about the meanings of positive and negative exponents. Others might require some calculator exploration of ordered pairs that satisfy the exponential equation $y = 10^x$.

a. 100 **b.** 10,000 **c.** 1,000,000

d. 0.01 **e.** −0.001 **f.** 3.45

g. −34.5 **h.** 345 **i.** 0.0023

2 Suppose that the sound intensity of a screaming baby was measured as 9.5 $\frac{\text{watts}}{\text{cm}^2}$. To calculate the equivalent intensity in decibels, 9.5 must be written as 10^x for some value of x.



a. Between which two integers does it make sense to look for values of the required exponent? How do you know?

b. Which of the two integer values in Part a is probably closer to the required power of 10?

c. Estimate the required exponent to the nearest hundredth. Then use your estimate to calculate a decibel rating for the loudness of the baby's scream.

d. Estimate the decibel rating for loudness of sound from a television set that registers intensity of 6.2 $\frac{\text{watts}}{\text{cm}^2}$.

Common Logarithms As you probably discovered in your work on Problems 1 and 2, it is not easy to solve equations like $10^x = 9.5$ or $10^x = 0.0023$—even by estimation. To deal with this very important problem, mathematicians have developed procedures for finding exponents. If $10^x = y$, then x is called the **base 10 logarithm** of y.

This definition of base 10 or *common logarithm* is usually expressed in function-like notation:

$$\log_{10} a = b \text{ if and only if } 10^b = a.$$

$\log_{10} a$ is pronounced "log base 10 of a." Because base 10 logarithms are so commonly used, $\log_{10} a$ is often written simply as $\log a$. Most scientific calculators have a built-in log function (LOG) that automatically finds the required exponent values.

3 Use your calculator to find the following logarithms. Then compare the results with your work on Problem 1.

a. log 100 **b.** log 10,000 **c.** log 1,000,000

d. log 0.01 **e.** log (−0.001) **f.** log 3.45

g. log (−34.5) **h.** log 345 **i.** log 0.0023

Summarize the Mathematics

In work on the problems of this investigation, you learned how physical measurements of sound intensity and acidity of a chemical substance are converted into the more familiar decibel and pH numbers. You also learned how the *logarithm* function is used in those processes.

a How would you explain to someone who did not know about logarithms what the expression $\log y = x$ tells about the numbers x and y?

b What can be said about the value of $\log y$ in each case below? Give brief justifications of your answers.

- **i.** $0 < y < 1$
- **ii.** $1 < y < 10$
- **iii.** $10 < y < 100$
- **iv.** $100 < y < 1{,}000$

Be prepared to explain your ideas to the class.

Check Your Understanding

Use your understanding of the relationship between logarithms and exponents to help complete these tasks.

a. Find these common (base 10) logarithms without using a calculator.

i. log 1,000 **ii.** log 0.001 **iii.** $\log 10^{3.2}$

b. Use the function $y = 10^x$, but not the logarithm key of your calculator, to estimate each of these logarithms to the nearest tenth. Explain how you arrived at your answers.

i. log 75 **ii.** log 750 **iii.** log 7.5

c. If the intensity of sound from a drag race car is 125 watts per square centimeter, what is the decibel rating of the loudness for that sound?

4 What do your results from Problem 3 (especially Parts e and g) suggest about the kinds of numbers that have logarithms? See if you can explain your answer by using the connection between logarithms and the exponential function $y = 10^x$.

5 Logarithms can be used to calculate the decibel rating of sounds, when the intensity is measured in $\frac{\text{watts}}{\text{cm}^2}$.

a. Use the logarithm feature of your calculator to rewrite 9.5 as a power of 10. That is, find x so that $9.5 = 10^x$.

b. Recall that if the intensity of a sound is $10^x \frac{\text{watts}}{\text{cm}^2}$, then the expression $10x + 120$ can be used to convert the sound's intensity to decibels. Use your result from Part a to find the decibel rating of the crying baby in Problem 2.

6 Assume the intensity of a sound $I = 10^x \frac{\text{watts}}{\text{cm}^2}$.

a. Explain why $x = \log I$.

b. Rewrite the expression for converting sound intensity readings to decibel numbers using $\log I$.

7 Use your conversion expression from Problem 6 to find the decibel rating of the television set in Problem 2 Part d.

Why Do They Taste Different? You may recall from your study of science that the acidity of a substance is described by its pH rating—*the lower its pH, the more acidic a substance is.* The acidity depends on the hydrogen ion concentration in the substance (in moles per liter). Some sample hydrogen ion concentrations are given below. Since those hydrogen ion concentrations are generally very small numbers, they are converted to the simpler pH scale for reporting.

8 Examine the table at the right.

a. Describe how hydrogen ion concentrations $[H^+]$ are converted into pH readings.

b. Write an equation that makes use of logarithms expressing pH as a function of hydrogen ion concentration $[H^+]$.

Substance	$[H^+]$	pH
Hand soap	10^{-10}	10
Egg white	10^{-9}	9
Sea water	10^{-8}	8
Pure water	10^{-7}	7
White bread	10^{-6}	6
Coffee	10^{-5}	5
Tomato juice	10^{-4}	4
Orange juice	10^{-3}	3

9 Use the equation relating hydrogen ion concentration and pH reading to compare acidity of some familiar liquids.

a. Complete a copy of the table at the right. Round results to the nearest tenth.

Substance	$[H^+]$ Proportion	pH Reading
lemonade	0.00501	
apple juice	0.000794	
milk	0.000000355	

b. Explain how your results tell which is more acidic—lemonade, apple juice, or milk.