NUMB3RS Activity: Changing Sines
Episode: “Counterfeit Reality”

Topic: Introduction to trigonometric graphs **Grade Level:** 10 - 12
Objective: To relate period, amplitude and phase shift to the graph of the basic sine function.
Materials: TI-83 Plus/TI-84 Plus graphing calculator
Time: 15 - 20 minutes

Introduction

When Charlie explains how he is looking for counterfeit money, he mentions that the engraving on the borders of paper money uses a technique called guilloche (pronounced “gee-YŌSH” – hard “g”). The overall design is very intricate but is produced using a combination of graphs of variations of the sine curve, called *sinusoids*. This activity uses elementary sine graphs to demonstrate how changes in the phase shift, amplitude, and period of the sine function affect its graph. In the form $y = a \sin [b(x + c)] + d$, these are the values of a, b, c, and d.

Discuss with Students

Students should be familiar with the sine function and the general shape of its graph. The teacher should also make sure students are familiar with the GRAPH and WINDOW functions on a graphing calculator (the TI-84 Plus is used in this activity). In addition, the class should be familiar with the term “radian,” since it is used in this activity. With each set of equations graphed, introduce the appropriate term for each of the changes to the original graph. In #3, this is *horizontal shift*. In #6, it is *amplitude*, in #10 *period*, and in #13 *vertical shift*. Time permitting (and if appropriate), after #10 introduce the formula for the period of $y = \sin[b(x)]$: period $= (2\pi/b)$ for radians, or $(360°/b)$ for degrees.

Student Page Answers:
1. *The graph appears to shift to the left by 1 and 2 units respectively. Explanations will vary. For example, when x = –1,* Y_2 *= sin(0)and when x = –2,* Y_2*= sin(0)so the graphs cross the x-axis at –1, and –2 respectively. A similar argument applies to each point on the graph.* **2.** *The graph appears to shift to the right by 1 and 2 units respectively.* **3.** *The graph appears to shift horizontally by (–c) units .Explanations will vary. For example, when x = –c, y = sin(x + c) = sin(0) = 0 and the point (–c, 0) is on the graph. A similar argument applies to each point on the graph.* **4.** *The graphs all have the same x-intercepts. The coefficients of 2 and 3 cause the range to double and triple, respectively.* **5.** *The ranges are 4 and 5 times as large respectively.* **6.** *They cross the x-axis at the same points as y = sin x, but the “height” is multiplied by a factor of “a” (even if 0 < a < 1).* **7.** *The graph reflects over the x-axis. Explanations will vary. For example, the reflection of any point (x, y) across the x-axis is (x, –y),* **8.** *The graph has the same amplitude, but there are two complete cycles over the same domain as the original equation.* **9.** *There are now three complete cycles (periods) compared to the original.* **10.** *This creates “b” periods over the original domain (if 0 < b < 1, the period increases). This number is used to measure frequency – see extensions).* **11.** *The graphs shift upward by 1 and 2 units respectively. Explanations will vary. For example, adding a constant to the y-coordinate of each point has the effect of raising the graph by that constant value.* **12.** *The graphs shift downward by 1 and 2 respectively.* **13.** *The graph shifts vertically by d units.*

Bro. Patrick Carney, De Paul Catholic High School, Wayne, NJ

Name: ________________________________ Date: _______________

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When Charlie discusses how to identify counterfeit paper money, he shows the delicate engraving used on the bills and explains that it is an application of a technique called guilloche (pronounced "gee-YŌSH" – hard "g"). This activity demonstrates how trigonometric functions can be used to generate figures that resemble guilloche.

The activity begins with $y = \sin(x)$ and then makes little changes in the equation to see how the graphs change.

On a TI-83 Plus/TI-84 Plus graphing calculator, set the [WINDOW] to: Xmin = –10, Xmax = 10, Xscl = 1, Ymin = –2, Ymax = 2, Yscl = 1. Be sure that the [MODE] is set to "Radian" and "Sequential". Then enter the equations below:

$$Y_1 = \sin(X)$$
$$Y_2 = \sin(X + 1)$$
$$Y_3 = \sin(X + 2)$$

Press [GRAPH] to view the graphs of these equations, and carefully observe the changes from one to the next.

1. Describe how the graph changes from the first to the third. Think about the function you are graphing and explain why this happens?

Turn off the graphs of Y_2 and Y_3, and then add the following:

$Y_4 = \sin(X – 1)$
$Y_5 = \sin(X – 2)$

2. Before graphing, predict the results. Then check your prediction against the graphs.

3. Generalize the effect on the graph of $y = \sin(x)$ if it is replaced by $y = \sin(x + c)$ for any number c. Explain your thinking.

Now, change the [WINDOW] to Ymin= –5 and Ymax = 5 (keep the other settings the same). Delete equations Y_4 and Y_5, then enter the following:

$Y_2 = 2\sin(X)$
$Y_3 = 3\sin(X)$

4. Graph these equations. What is the same in all three graphs? Why? How do the coefficients of 2 and 3 change the graphs?

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Now add the following:

Y_4 = 4sin(X)

Y_5 = 5sin(X)

5. Before graphing, predict the effect and explain why you think your prediction is reasonable. Then check your prediction against the graphs.

6. Generalize the effect on the graph of $y = \sin(x)$ if it is replaced by $y = a \sin(x)$ for a positive number a.

7. Suppose $a < 0$. How does that change the answer to #6? Explain your reasoning. Check using the calculator.

Delete all equations except Y_1. Change the WINDOW to Xmin = 0, Xmax = 10, Ymin = –2, and Ymax = 2. Then enter: Y_2 = sin(2X).

8. Graph these equations. Describe the effect of multiplying X by 2.

Now add Y_3 = sin(3X).

9. Before graphing, predict the effect, then check your prediction against the graphs.

10. Generalize the effect on the graph of $y = \sin(x)$ if it is replaced by $y = \sin(bx)$ ($b \neq 0$).

Delete all equations except Y_1. Change the WINDOW to Xmin = -10, Xmax = 10, Ymin = –5, Ymax = 5, then enter the following:

Y_2 = sin(X) + 1

Y_3 = sin(X) + 2

11. Describe the effect of adding the 1 and the 2 to sin(X). Why does this make sense?

12. Change the equations in the previous question to subtract 1 and 2 rather than add. Before graphing, predict the effect, and then check your prediction against the graphs.

13. Generalize the effect on the graph of $y = \sin(x)$ if it is replaced by $y = \sin(x) + d$ for any number d.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

With a magnifying glass (if you need it), take a very close look at the designs around the edge of a $1 or $10 bill (these have the most). Look at the “spider webs” along the border. This is an example of the guilloche process. Follow a single line closely and try to identify the shape of one of the sine curves from this activity. Obviously, the overall design is a mixture of many of these curves, called *sinusoids*. Try to duplicate even a small part of it, either by hand or with your calculator. It quickly becomes clear why guilloche is used to make counterfeiting very difficult.

Related Topics

- In the general form $y = a \sin[b(x + c)] + d$, “a” is called the amplitude and “b” the frequency. Radio signals are broadcast in frequency ranges called AM and FM. Find out what AM and FM stand for and how they relate to trigonometry.

- If you have a sound system, it probably has an amplifier. What exactly does the word “amplify” mean? Research how this relates to the “a” in the general form equation above and the corresponding graph.

- An oscilloscope is an instrument that converts sound into complex combinations of the graphs in this activity. Your school’s physics department may be able to give you access to one, along with some tuning forks. Strike two tuning forks that are an octave apart and watch the graphs. (The tuning fork gives a pure tone rather than the complicated combinations in ordinary music.) Experiment by keeping the note the same and changing the volume. Relate these experiments to the general form equation above.

- Many computer music players have “skins” that are actually oscilloscopes. Listen to some music and observe the display with a more “mathematical eye.”

- Some of these designs are made using polar coordinates. Set your calculator to **Polar** mode and experiment with graphing some of these same equations.

Additional Resources

- For more information about guilloche with more mathematical connections, see: **http://mathworld.wolfram.com/GuillochePattern.html**

- For examples of banknotes from around the world, along with a related link that shows how the very intricate and valuable Fabergé eggs are made, see: **http://www.maa.org/editorial/mathgames/mathgames_02_09_04.html**

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